PHYSICAL REVIEW E **70**, 015603(R) (2004)

Theory of incoherent optical solitons: Beyond the mean-field approximation

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(Received 9 December 2003; published 27 July 2004)

We present a general theory of partially coherent optical solitons in slow-responding nonlinear media that takes into account intensity fluctuations of the light sources generating such solitons. If intensity fluctuations of the source are negligible, the theory reduces to the previously reported mean-field theory of partially coherent solitons. However, when such fluctuations are significant, our theory shows that the properties of partially coherent solitons in saturable nonlinear media can be qualitatively different from those predicted by the mean-field theory.

DOI: 10.1103/PhysRevE.70.015603 PACS number(s): 42.65.Tg, 42.25.Kb, 42.65.Jx

The behavior of fluctuating fields coupled via a nonlinear interaction is a rich and challenging subject of considerable interest in the areas of physics as diverse as hydrodynamic turbulence [1] and the theory of phase transitions [2]. In the domain of optics, the interest in this topic has recently been reinforced by experiments on the generation of solitons using partially coherent [3], and even nearly incoherent, thermal-like [4] sources of light.

In general, a theoretical description of nonlinearly coupled statistical fields in terms of appropriate correlation functions is a formidable task, because one has to deal with an infinite hierarchy of evolution equations for the correlation functions of all orders (see, for example [1]). In certain cases, however, the use of the *mean-field* approximation (MFA), which amounts to neglecting any effects associated with the fourth- and higher-order field correlations, can be justified. Indeed, the MFA has been quite successful in understanding phase transitions [2], Bose–Einstein condensation [5], and statistical wave propagation in nonlinear optical media [6–9]. In the optical context, the MFA can be justified when the response time of a nonlinear medium is much longer than the typical correlation time associated with phase fluctuations across the wave front. Within the framework of the mean-field approximation, the nonlinear refractive index of the medium is assumed to depend only on the average intensity of an optical beam. This crucial assumption has made it possible to put forward several equivalent formulations of the mean-field theory of partially coherent solitons [7–9].

However, the mean-field theory of partially coherent solitons does not take into account intensity fluctuations of the source generating such solitons. Consequently, some of the predictions of the mean-field theory concerning white-light solitons, produced by thermal-like sources with large intensity fluctuations [4,10], can be questionable. Thus, it is important to determine how *intensity fluctuations in the source plane* affect the salient properties of the optical solitons generated by partially coherent sources.

In this Rapid Communication, we formulate a theory of partially coherent optical solitons that goes *beyond* the mean-field approximation. Such a theory can be formulated either using the evolution equation for the cross-spectral density function of the optical field or, equivalently, using the self-consistent, multimode waveguide approach similar to that of Ref. [8]. The key assumption of the new theory is that the nonlinear response to a fluctuating optical field of a slow-responding medium, whose response time is much longer than the typical correlation time associated with phase fluctuations of the field, can be characterized by the *time-averaged* nonlinear refractive index. This approach generalizes the previously developed mean-field theory of partially coherent spatial solitons and is capable of taking into account the contribution of source intensity fluctuations to the nonlinear response of the medium. For this reason, our theory is expected to provide a more accurate *quantitative* description of the white-light solitons generated in a recent experiment [4], compared with that provided by the mean-field theory of Ref. [10]. To demonstrate the influence of source intensity fluctuations on the qualitative features of spatial solitons, we compare the intensity profiles, as well as the spatial coherence lengths, of the solitons generated by fluctuating, quasi-monochromatic laser light with those produced by spectrally filtered thermal light.

We consider a statistically stationary source which generates a partially coherent polychromatic light beam that propagates in a nonlinear medium close to the z axis. The electric field $\mathbf{E}$ of the beam is assumed to be linearly polarized such that $\mathbf{E}(\boldsymbol{\rho},z,t)=\hat{\mathbf{x}}U(\boldsymbol{\rho},z,t)e^{ik_0z}$, where $k_0=\omega_0 n_0/c$, with n_0 being the linear refractive index of the medium at the carrier frequency ω_0. The slowly varying envelope function $U(\boldsymbol{\rho},z,t)$ obeys the paraxial wave equation,

$$i\frac{\partial U}{\partial z}+\frac{1}{2k_0}\nabla_\perp^2 U=\frac{n_0}{k_0c^2}\langle n_{nl}(I)\rangle\frac{\partial^2 U}{\partial t^2}, \tag{1}$$

where the angle brackets denote averaging over the ensemble of statistical realizations of the beam which is equal to the time averaging due to ergodicity [11].

The second-order statistical properties of a statistically stationary optical field are described in terms of the cross-spectral density $W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z,\omega)$. The latter is related to the second-order correlation function $\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z,\tau)$ in the spacetime domain via the generalized Wiener-Khintchine theorem [11, Sec. 2.4.4],

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z,\omega) = \int_{-\infty}^{\infty} d\tau \Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z,\tau)e^{i\omega\tau}, \qquad (2)$$

where $\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z,\tau)\equiv\langle U^*(\boldsymbol{\rho}_1,z,t+\tau)U(\boldsymbol{\rho}_2,z,t)\rangle$. It follows from Eqs. (1) and (2), and the definition of Γ that the cross-spectral density obeys the wave equation in the form

$$i\frac{\partial W}{\partial z} + \frac{1}{2k_0}(\nabla_{\perp 1}^2 - \nabla_{\perp 2}^2)W + \frac{n_0\omega^2}{k_0c^2}[\langle n_{nl}(I_1)\rangle - \langle n_{nl}(I_2)\rangle]W = 0, \qquad (3)$$

where $I_j\equiv I(\boldsymbol{\rho}_j,z)$ denotes instantaneous intensity $(j=1,2)$. The ensemble-averaged nonlinear refractive index in Eq. (1) can be expressed as

$$\langle n_{nl}(I)\rangle = \int_0^{\infty} dI n_{nl}(I)P(I), \qquad (4)$$

where $P(I)$ is the probability distribution of the instantaneous intensity of the beam.

Equations (3) and (4) are the key result of our theory expressed in the language of correlation functions. If the cross-spectral density remains the same in every transverse plane along the z axis, the beam represents a spatial soliton supported by the medium. Alternatively, one can formulate our theory by extending the self-consistent multimode waveguide approach of Ref. [8]. To this end, we recall that the cross-spectral density can be represented as a Mercer-type series by the expression [11, Sec. 4.7]

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z,\omega) = \sum_s \lambda_s(\omega)\psi_s^*(\boldsymbol{\rho}_1,z,\omega)\psi_s(\boldsymbol{\rho}_2,z,\omega). \qquad (5)$$

Here $\lambda_s(\omega)\geqslant 0$ denotes a spectral weight of each mode in the expansion and $\psi_s(\boldsymbol{\rho},z,\omega)$ is a modal function, with s being the set of indices labeling the modes. On substituting from Eq. (5) into Eq. (3), one obtains the wave equation for each mode in the form,

$$i\frac{\partial \psi_s}{\partial z} + \frac{1}{2k_0}\nabla_{\perp}^2\psi_s + \frac{n_0\omega^2}{k_0c^2}\langle n_{nl}(I)\rangle\psi_s = 0. \qquad (6)$$

The modal functions of partially coherent solitons have the form $\psi_s(\boldsymbol{\rho},z,\omega)=u_s(\boldsymbol{\rho},\omega)\exp[i\beta_s(\omega)z]$, where $\beta_s(\omega)$ is a mode propagation constant.

The theory of the propagation of statistical waves in nonlinear media that we have presented so far remains incomplete until one provides a recipe for finding the probability distribution $P(I)$ for the intensity of the beam in the nonlinear medium, which can be a formidable task in general. Fortunately, however, the intensity statistics of the beam can be determined in the two cases of particular importance, as far as realistic partially coherent light sources are concerned.

Laserlike sources: Consider a partially coherent source generated by a well-stabilized multimode laser. The probability distribution $P(I)$ in the source plane can then be approximated by a delta function, i. e., $P(I_s)\propto\delta(I_s-\langle I_s\rangle)$, where I_s is the intensity at the source [11, Sec. 11.8]. Further, provided the characteristic longitudinal correlation length of the field, $L_c\simeq k_0\sigma_c^2$, where σ_c is the transverse coherence length of the source, is much smaller than a typical distance over which the nonlinear interaction of the field U with the fluctuations of the nonlinear refractive index takes place [6], we can safely neglect any changes in the statistics of the field intensity on its propagation and assume that $P(I)\propto\delta(I-\langle I\rangle)$. It then follows at once from Eq. (4) that for a laserlike source $\langle n_{nl}(I)\rangle=n_{nl}(\langle I\rangle)$, and the present theory reduces to the mean-field theory developed in Refs. [6–9].

Thermal-like sources: Consider now light generated by a thermal-like source such as, for example, a light bulb or the LED employed in the experiment [4]. In this case, intensity fluctuations of the source can be very large; typically the standard deviation σ_I of such fluctuations is $\sim\langle I_s\rangle$. At the same time, the longitudinal correlation length of the field generated by such a source is of the order of a wavelength λ. If the power of the source is not too large, the typical interaction length of the field with fluctuations $\delta n_{nl}=n_{nl}-\langle n_{nl}\rangle$ is much greater than any wavelength in the spectrum of the field. Hence, we can assume that the probability distribution, of the intensity of the field in any transverse plane of the beam has the same functional form as that in the source plane. For a thermal source, $P(I)\propto e^{-I/\langle I\rangle}$ [11, Sec. 11.8]. It then follows from Eq. (4) that

$$\langle n_{nl}(I)\rangle = \int_0^{\infty}\frac{dI}{\langle I\rangle}n_{nl}(I)\exp\left(-\frac{I}{\langle I\rangle}\right), \qquad (7)$$

where the average intensity is related to the cross-spectral density as

$$\langle I(\boldsymbol{\rho},z)\rangle = \int_0^{\infty} d\omega W(\boldsymbol{\rho},\boldsymbol{\rho},z,\omega). \qquad (8)$$

Equations (3), (7), and (8), along with the condition $\partial W/\partial z=0$, describe theoretically the white-light solitons, supported by a medium with a given nonlinear refractive index $n_{nl}(I)$. Equivalently, white-light solitons existing in such a medium are composed of the modes determined by a self-consistent solution of Eqs. (6)–(8). It follows from Eq. (7) that, since in general $\langle n_{nl}(I)\rangle\neq n_{nl}(\langle I\rangle)$ [12], spatial solitons generated by a strongly fluctuating source are trapped by the self-induced waveguide whose shape differs from that predicted by the mean-field theory.

To illustrate the impact of source intensity fluctuations on the qualitative and quantitative features of partially coherent solitons, we compare the solitons generated by a thermal source with those produced by a quasimonochromatic, laserlike source. We focus on the intensity distribution as well as on the spatial coherence length of the generated solitons and assume that the light from the thermal source is made quasimonochromatic by transmitting it through a narrow-band spectral filter centered at the frequency ω_0 [13]. The spectral weight $\lambda_s(\omega)$ of each mode ψ_s can then be approximated as $\lambda_s(\omega)=\lambda_s\delta(\omega-\omega_0)$, and the averaged total intensity of the soliton can be written as

$$\langle I(\rho)\rangle = \sum_s \lambda_s|u_s(\boldsymbol{\rho},\omega_0)|^2. \qquad (9)$$

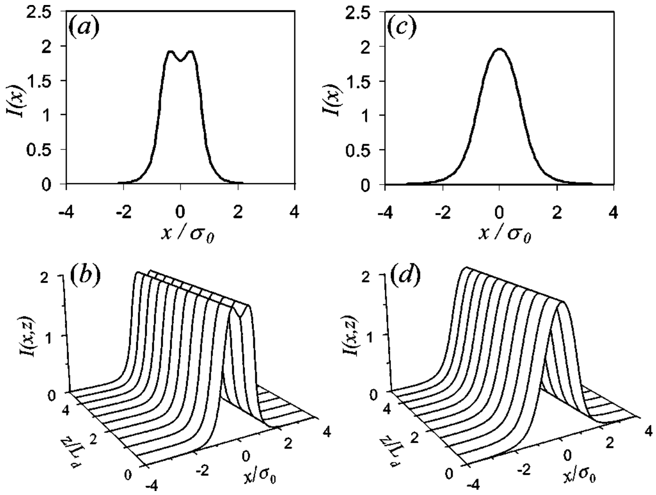

FIG. 1. (a) Intensity profile and (b) its evolution for a two-mode partially coherent soliton in the medium with the threshold-type nonlinearity predicted by the mean-field theory. Curves (c) and (d) show the corresponding results obtained using the generalized theory. Transverse spatial scale σ_0 is chosen such that $k_0^2\sigma_0^2\Delta n/n_0=5$. Intensities are normalized by setting $I_{th}=1$. The mode amplitudes in the mean-field limit are $\lambda_1=1.778$ and $\lambda_2=0.819$; those predicted by the generalized theory are $\lambda_1=1.966$ and $\lambda_2=0.440$.

We have performed our numerical simulations for media with two types of nonlinearity: a threshold-type nonlinearity for which $n_{nl}(I)=0$ for $I\leqslant I_{th}$, and $n_{nl}(I)$ takes a constant value Δn for $I>I_{th}$, and a photorefractive nonlinearity with $n_{nl}(I)=\Delta n(I/I_{th})/(1+I/I_{th})$. It follows from Eq. (7) that solitons produced by a filtered thermal source in media with the threshold-type and photorefractive nonlinearities are trapped by self-induced waveguides with the nonlinear refractive indices of the form

$$\langle n_{th}(I)\rangle = \Delta n\ \exp(-I_{th}/\langle I\rangle) \qquad (10)$$

and

$$\langle n_{ph}(I)\rangle = \Delta n[1-(I_{th}/\langle I\rangle)e^{I_{th}/\langle I\rangle}\ \mathrm{Ei}(I_{th}/\langle I\rangle)], \qquad (11)$$

respectively. Here $\mathrm{Ei}(x)=\int_x^\infty dz\ e^{-z}/z$ is the exponential integral function.

We have determined the modal functions $\psi_s(x,z,\omega_0)$ of $(1+1)D$ solitons in such nonlinear media by solving Eqs. (6) and (9)–(11) self-consistently, following the procedure described in Ref. [14]. Our numerical calculations were performed in the soliton units: $x'=x/\sigma_0$, $z'=z/L_d$, and $u_s'(x')=\sqrt{\lambda_s}u_s(x)$, where $L_d=k_0\sigma_0^2$ is a characteristic diffraction length and σ_0 is a transverse spatial scale. Figures 1 and 2 show the results for the threshold-type nonlinearity [15] for solitons consisting of two and four modes, respectively. The left column shows the results obtained in the MFA valid for laserlike sources. It is evident that the intensity profiles of the solitons generated by laserlike and thermal sources of the same power are qualitatively different. It is also seen from Figs. 1 and 2 that the two- and four-mode solitons maintain their shapes on propagation over several diffraction lengths, which attests to their stability at least over such distances. In the case of photorefractive nonlinearity, the solitons predicted by the MFA and our theory have nearly the same intensity profile, even though the corresponding modal functions are different.

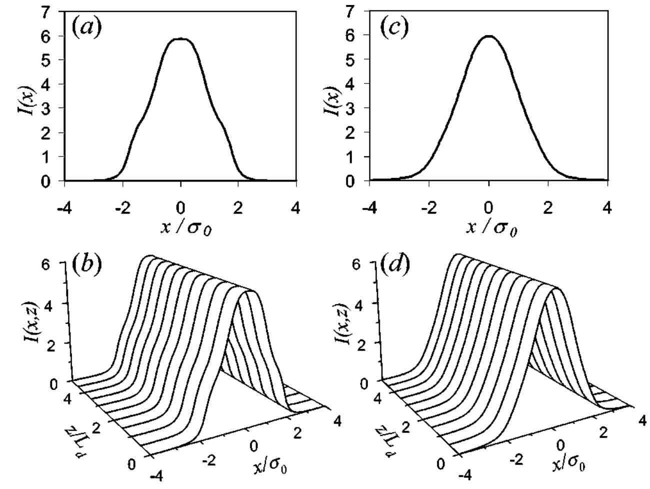

FIG. 2. Same as in Fig. 1, except that the corresponding partially coherent soliton is composed of four modes with $\lambda_1=5.461$, $\lambda_2=1.337$, $\lambda_3=0.600$, and $\lambda_4=0.316$. In the mean-field limit, $\lambda_1=5.445$, $\lambda_2=0.252$, $\lambda_3=0.423$, and $\lambda_4=0.473$.

To illustrate how the coherence properties of spatial solitons are affected by source intensity fluctuations, we compare the spatial coherence length l_c of the solitons generated by thermal and laserlike sources. We define l_c as $l_c(x)=\int_{-\infty}^{\infty} d(\Delta x)|\mu(x,x+\Delta x)|$, where $\mu(x,\Delta x)=W(x,x+\Delta x)/[\langle I(x)\rangle\langle I(x+\Delta x)\rangle]^{1/2}$ is the degree of coherence of the soliton. Figure 3 shows the spatial coherence lengths of the four-mode solitons generated by thermal- and laserlike sources in the threshold-type [Fig. 3(a)] and photorefractive [Fig. 3(b)] nonlinear media as functions of x/σ_0. In the insets to Figs. 3(a) and 3(b), the behavior of the corresponding nonlinear refractive indices is displayed. It is seen from Fig. 3 that for any value of x, the magnitude of the coherence length of the soliton generated by the thermal-like source is smaller than that of its counterpart produced by the laserlike

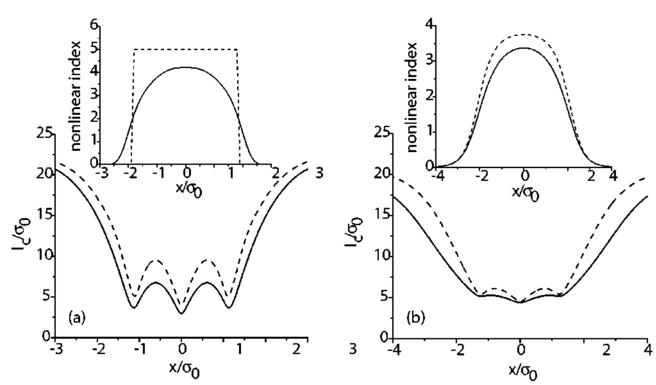

FIG. 3. Spatial coherence length as a function of x/σ_0 for the four-mode solitons generated by thermal (solid line) and laserlike (dashed line) sources in (a) threshold-type and (b) photorefractive nonlinear media. The insets show the averaged nonlinear refractive index. In the case of saturable nonlinearity, we choose $I_{th}=2$ and $k_0^2\sigma_0^2\Delta n/n_0=4$.

source. It can also be inferred from Fig. 3 that even though the self-induced waveguides of the solitons generated by thermal- and laserlike sources in the photorefractive medium have a very similar shape, the spatial distributions of the coherence lengths of the two types of solitons may significantly differ as a result of different spatial distributions of the soliton modes. This conclusion is confirmed by numerical simulations.

In summary, we have developed a general theory of partially coherent optical solitons in slow-responding nonlinear media that takes into account source intensity fluctuations. We have applied this theory to laserlike sources with negligible intensity fluctuations and to thermal-like sources with very large intensity fluctuations. Our results demonstrate that the properties of partially coherent solitons produced by the latter sources in slow-responding nonlinear media can be qualitatively different from those predicted by the mean-field theory. In particular, even when the soliton shape predicted by the MFA and by our theory is nearly the same, as is the case in the experimentally important photorefractive nonlinearity, the two theories predict different soliton coherence properties. We expect this work to motivate further studies toward *quantitative* comparison of theoretically predicted and experimentally measured characteristics of the optical solitons generated by thermal sources.

This research was supported by the U.S. Air Force under Grant No. F49260-03-1-0138, by the Department of Energy under Grant No. FR-FG02-2ER45992, by the National Science Foundation under Grant No. ECS-0320816, and by DARPA under Grant No. MDA-972011043.

[1] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics; Mechanics of Turbulence* (MIT Press, Cambridge, Massachusetts, 1971).

[2] L. Reichl, *A Modern Course in Statistical Physics* (Wiley, New York, 1998), Chap. 8.

[3] M. Mitchell, Z. Chen, M. F. Shih, and M. Segev, Phys. Rev. Lett. **77**, 490 (1996).

[4] M. Mitchell and M. Segev, Nature (London) **387**, 880 (1997).

[5] A. Griffin, D. W. Snoke, and S. Stringari, *Bose-Einstein Condensation* (Cambridge University Press, Cambridge, 1995), Chap. 1.

[6] G. A. Pasmanik, Sov. Phys. JETP **39**, 234 (1974).

[7] D. N. Christodoulides, T. H. Coskun, M. Mitchell, and M. Segev, Phys. Rev. Lett. **78**, 646 (1997).

[8] M. Mitchell, M. Segev, T. H. Coskun, and D. N. Christodoulides, Phys. Rev. Lett. **79**, 4990 (1997).

[9] V. V. Shkhunov and D. Z. Anderson, Phys. Rev. Lett. **81**, 2683 (1998).

[10] H. Buljan, M. Segev, M. Soljacic, N. K. Efremidis, and D. N. Christodoulides, Opt. Lett. **28**, 1239 (2003); See, also, H. Buljan, A Siber, M. Soljacic, T. Scwartz, M. Segev, and D. N. Christodoulides, Phys. Rev. E **68**, 036607 (2003), and references therein.

[11] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995).

[12] The statistics of intensity fluctuations play no role for a Kerr medium with $n_{nl}(I)=n_2 I$, because $\langle n_{nl}(I)\rangle = n_{nl}(\langle I\rangle)$, regardless of the statistics. One can understand qualitatively this feature of the theory by recalling that any nonlinear medium behaves as Kerr-like in the low-intensity limit, and the source intensity fluctuations are negligible in this limit.

[13] In this work, we focus on the role of intensity fluctuations of a thermal source. In general, such a source, if not filtered, is polychromatic, which results in the dependence of the soliton spatial coherence length on frequency. This effect was investigated under the MFA in Ref. [10].

[14] N. M. Litchinitser, W. Królikowski, N. N. Akhmediev, and G. P. Agrawal, Phys. Rev. E **60**, 2377 (1999).

[15] Solitons produced by such sources are trapped by a self-induced waveguide with the nonlinear refractive index $n_{nl}(\langle I\rangle)=0$ if $\langle I\rangle < I_{th}$, and $n_{nl}(\langle I\rangle)=\Delta n$ otherwise. To study numerically the formation and propagation of the solitons with such a discontinuous nonlinearity, we have replaced the stepwise nonlinear refractive index by the smooth approximation $n_{nl}(\langle I\rangle)=\Delta n\{1+\tanh[N(\langle I\rangle - I_{th})]\}/2$, where $N \gg 1$ is a large numerical parameter chosen to fit the stepwise refractive index profile to desired accuracy. In our calculations, we let $N=50$.